# MOTION IN TWO DIMENSION

## 10. SCALARS AND VECTORS

Some quantities can be deseribed by single number. For e.g.: Mass, time, distance, speed. One piece of information is enough to describe them fully. These are called **SCALAR** quantities.

To tell someone how to get to Lakshya from some location, one piece of information is not enough. To describe this fully, both distance and displacement are required. Quantities which require both magnitude and direction to describe a situation fully are known as **VECTOR**. For e.g.: displacement, velocity

The vectors are denoted by putting an arrow over the symbols representing them.

For e.g.: AB vector can be represented by $\overrightarrow{AB}$

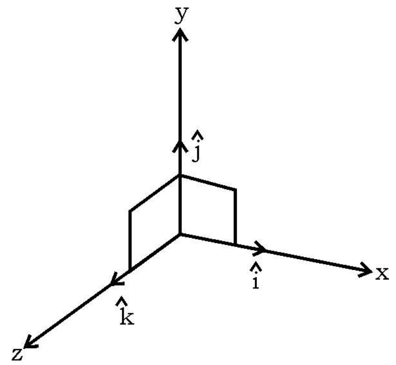
### 10.1 Unit vector

A unit vector has a magnitude of one and so it really gives just the direction of the vector.

A unit vector can be found by dividing the original vector by its magnitude

$$\hat{a} = \frac{\vec{a}}{|a|}$$

unit vectors along different co–ordinate axis




### 10.2 Addition, subtraction and scalar multiplication of vectors

Suppose, we have two vectors

$$\vec{r}_1 = a_1\hat{i} + b_1\hat{j}$$

$$\vec{r}_2 = a_2\hat{i} + b_2\hat{j}$$

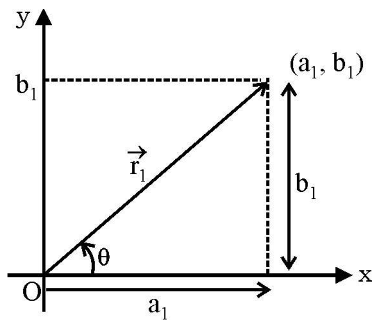
Then, $\vec{r}_1 + \vec{r}_2 = (a_1 + a_2)\hat{i} + (b_1 + b_2)\hat{j}$

$$\vec{r}_1 - \vec{r}_2 = (a_1 - a_2)\hat{i} + (b_1 - b_2)\hat{j}$$

Multiplication of a vector by scalar quantity.

$$c\vec{r}_1 = c(a_1\hat{i} + b_1\hat{j}) = ca_1\hat{i} + cb_1\hat{j}$$

Representation of $\vec{r}_1$ on the co–ordinate axis




magnitude and direction of $\vec{r}_1$

Magnitude of $\vec{r}_1 (|\vec{r}_1|) = \sqrt{a_1^2 + b_1^2}$

direction of $\vec{r}_1$

$$\tan\theta = \frac{b_1}{a_1} = \frac{\text{component y-axis}}{\text{component along x-axis}}$$

$$\theta = \tan^{-1}\left(\frac{b_1}{a_1}\right)$$

### 10.3 Parallel vectors

Two vectors are parallel if and only if they have the same direction. When any vector is multiplied by a scalar, a vector parallel to the original vector is formed.

If $\vec{b} = k\vec{a}$ then $\vec{b}$ and $\vec{a}$ are parallel vector. In general to find if two vectors are parallel or not we must find their unit vectors.

## 10.4 Equality of vectors

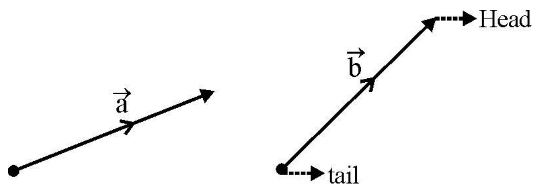
Two vectors (representing two values of the same physical quantity) are called equal if their magnitudes and directions are same.

For e.g. $(3\hat{i}+4\hat{j})\text{m}$ and $(3\hat{i}+4\hat{j})\text{m/s}$

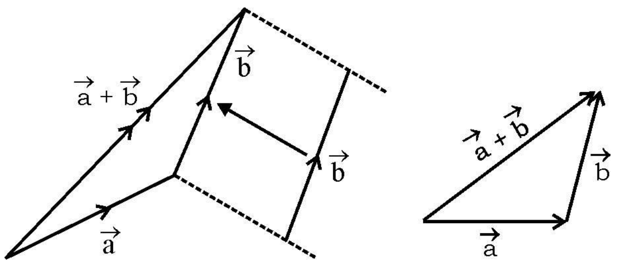
Cannot be compared as they represent two different physical quantities.

## 10.5 Addition of vectors

When two or more vectors are added, the answer is called the resultant. The resultant of two vectors is equivalent to the first vector followed immediately by the second vector.

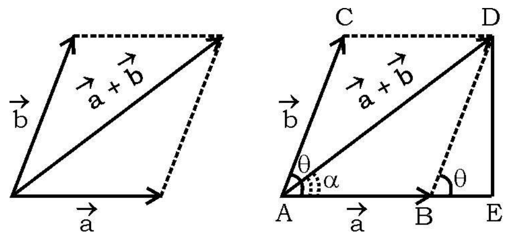



To find the resultant of vectors $\vec{a}$ and $\vec{b}$, the tail of vector $\vec{b}$ must join to the head of vector $\vec{a}$. The resultant $\vec{a}+\vec{b}$ is the direct vector from the tail of vector $\vec{a}$ to the head of vector $\vec{b}$.



**This is known as triangle rule of vector addition**

Another way is **parallelogram rule of vector addition** on this we draw vectors $\vec{a}$ and $\vec{b}$, with both the tails co–inciding. Taking these two odjacent sides we complete the parallelogram. the diagonal through the common tails gives the sum of two vectors.




Finding magnitude of $\vec{a}+\vec{b}$ and its direction

$|AD|^2 = AE^2 + ED^2$

$AE = |a| + |b \cos\theta|$

$ED = b \sin\theta$

$AD^2 = a^2 + b^2 \cos^2\theta + 2ab\cos\theta + b^2 \sin^2\theta$

$AD^2 = a^2 + b^2 + 2ab\cos\theta$

$AD = \sqrt{a^2 + b^2 + 2ab\cos\theta}$

where, θ is the angle contained between $\vec{a}$ and $\vec{b}$

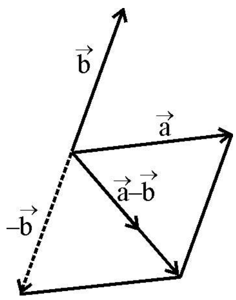
$\tan\alpha = \dfrac{ED}{AE} = \dfrac{b\sin\theta}{a + b\cos\theta}$

where α is the angle which the resultant makes with + x axis

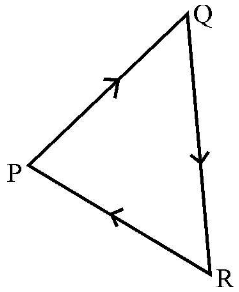
**Subtraction of vectors :**

Let $\vec{a}$ and $\vec{b}$ be two vectors. We define $\vec{a}-\vec{b}$ as sum of vectors $\vec{a}$ and the vectos $\left(-\vec{b}\right)$.

or, $\vec{a}+\left(-\vec{b}\right)$

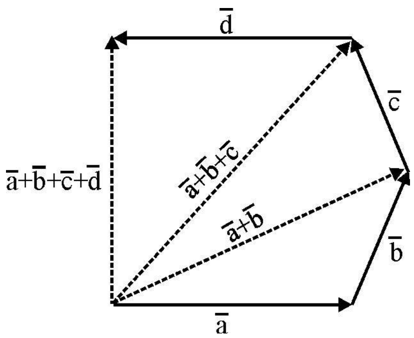


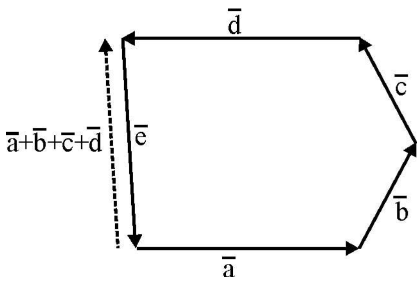
**Zero vector**




In this triangle $\overrightarrow{PQ}+\overrightarrow{QR}+\overrightarrow{PR}$ must be equal to zero as the overall journey results in a return to the starting point.

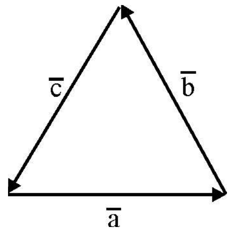
$$\overrightarrow{PQ}+\overrightarrow{QR}+\overrightarrow{RP}=\vec{O}$$



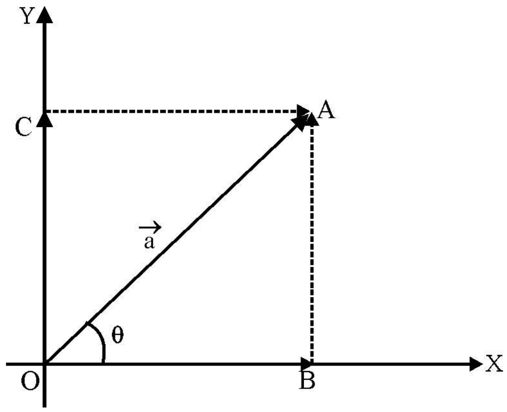

$$\bar{a}+\bar{b}+\bar{c}+\bar{d}+\bar{e}=0$$

$$\bar{e}=-(\bar{a}+\bar{b}+\bar{c}+\bar{d})$$




$$\bar{a}+\bar{b}+\bar{c}=0$$

**Resolution of vectors**




$$\overrightarrow{OA}=\vec{a}$$

By vector addition rule,

$$\overrightarrow{OA}=\overrightarrow{OB}+\overrightarrow{OC}$$

$$\left|\overrightarrow{OB}\right| = a\cos\theta$$

$$\left|\overrightarrow{OC}\right| = a\sin\theta$$

If $\hat{i}$ and $\hat{j}$ denote vectors of unit magnitude along OX and along OY respecively, we get

$$\overrightarrow{OB} = a\cos\theta\ \hat{i}$$

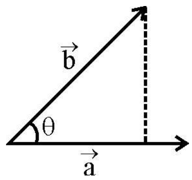
$$\overrightarrow{OC} = a\sin\theta\ \hat{j}$$

$$\vec{a}=(a\cos\theta)\hat{i}+(a\sin\theta)\hat{j}$$

Multiplication of 2 vectors

Scalar product | Vector product

### 10.6 Dot product or scalar product of two vector




$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

If, $\theta = 0° \Rightarrow \vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|$

$\theta = 90° \Rightarrow \vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}| \cos 90° = 0$

$\hat{i}\cdot\hat{i} = |\hat{i}||\hat{i}|\cos 0° = i^2 \times 1 = 1$

Similarly, $\hat{j}\cdot\hat{j} = 1$ $\quad\hat{k}\cdot\hat{k} = 1$

$\hat{i}\cdot\hat{j} = |\hat{i}||\hat{j}|\cos 90° = 1\times 1\times 0 = 0$

Similarly, $\hat{j}\cdot\hat{k} = 0$ and $\hat{k}\cdot\hat{i} = 0$

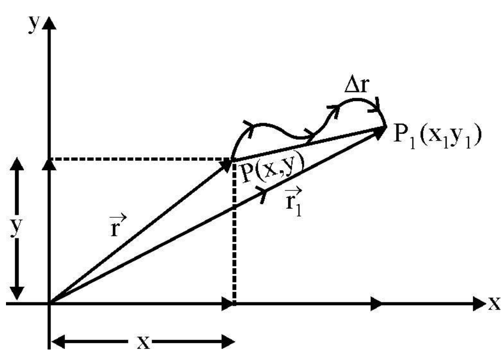
The dot product is commutative and distributive

$$\boxed{\begin{aligned}&\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}\\ &\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}\end{aligned}}$$

## 11. MOTION IN 2D (PLANE)

### 11.1 Position vector and Displacement

The position vector $\vec{r}$ of a particle P located in a plane with referecne to the origin of an xy–coordinate system is given by




$$\vec{r} = x\hat{i} + y\hat{j}$$

Suppose the particle moves along the path as shown to a new position $P_1$ with the position vector $\vec{r}$

$\vec{r}_1 = x_1\hat{i} + y_1\hat{j}$

change in position = displacement

$= \vec{r}_1 - \vec{r} = (x_1\hat{i} + y_1\hat{j}) - (x\hat{i} + y\hat{j})$

$= (x_1 - x)\hat{i} + (y_1 - y)\hat{j}$

$= \Delta x\hat{i} + \Delta y\hat{j}$

from above figure we can see that

$\vec{r} + \Delta\vec{r} = \vec{r}_1$ or $\Delta\vec{r} = \vec{r}_1 - \vec{r}$

(triangle law of vector addition)

### 11.2 Average velocity

$$\vec{v}_{av} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\Delta x\hat{i} + \Delta y\hat{j}}{\Delta t}$$

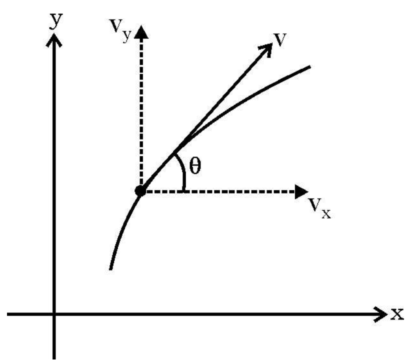
$$v_{avg} = v_x\hat{i} + v_y\hat{j}$$

**Note :** Direction of the average velocity is same as that of $\Delta\vec{r}$

### 11.3 Instantaneous velocity

$$\vec{v} = \lim_{\Delta t\to 0}\frac{\Delta v}{\Delta t} = \frac{d\vec{r}}{dt}$$

$$\vec{v} = v\times\hat{i} + v_y\hat{j}$$




where, $v_x = \dfrac{dx}{dt}$ and $v_y = \dfrac{dy}{dt}$

$$|\vec{v}| = \sqrt{v_x^2 + v_y^2}$$

$$\tan\theta = \frac{v_y}{v_x} \text{ or } \theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

**Note :** The direction of instantaneous velocity at any point on the path of an object is tangent to the path at that point and is in the direction of motion.

**11.4 Averge acceleration**

$$\vec{a}_{avg} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\Delta v_x}{\Delta t}\hat{i} + \frac{\Delta v_y}{\Delta t}\hat{j}$$

$$\vec{a}_{avg} = a_x\hat{i} + a_y\hat{j}$$

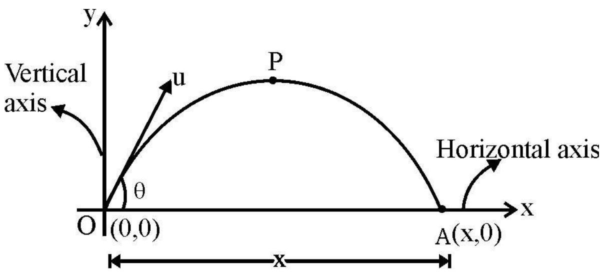
**11.5 Instantaneous acceleration**

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j}$$

$$\vec{a} = \vec{a}_x\hat{i} + a_y\hat{j}$$

## 12. PROJECTILE MOTION

When a particle is projected obliquely near the earth surface, it moves simultaneously in horizontal and vertical directions. Motion of such a particle is called projectile motion.




In this case a particle is projected at an angle θ with an initial velocity u. For this particular case we will calculate the following :

(a) time taken to reach A from O

(b) horizontal distance covered (OA)

(c) max$^m$ height reached during the motion

(d) velocity at any time 't' during the motion

| **Horizontal axis** | **Vertical axis** |
|---|---|
| $u_x = u\cos\theta$<br>$a_x = 0$<br>(In the absence of any external force $a_x$ will be assumed to be zero) | $u_y = u\sin\theta$<br>$a_y = -g$<br>$s_y = u_y t + 1/2\, a_y t^2$<br>$0 - 0 = u\sin\theta\, t - 1/2 g t^2$<br>$\boxed{T = \frac{2u\sin\theta}{g} = \frac{2u_y}{g}}$ |
| $s_x = u_x t + 1/2 a_x t^2$<br>$x - 0 = u\cos\theta\, t$<br>$x = u\cos\theta \times 2u_y/g$<br>$x = \frac{2u^2\cos\theta\sin\theta}{g}$<br>$\boxed{R = \frac{u^2\sin 2\theta}{g}}$<br>($\because$ 2 cos θ sin θ = sin 2θ) horizontal distance covered is known as **Range** | $v_y = u_y + a_y t$<br>$\boxed{v_y = u\sin\theta - gt}$<br>It depends on time 't'<br>It in not constant<br>It's magnitude first decreases becomes zero and then increases. |
| $v_x = u_x + a_x t$<br>$\boxed{v_x = u\cos\theta}$<br>It is independent of t<br>It is constant | **maximum height** obtained by the particle<br>Method 1 : using time of ascent<br>$s_y = u_y t_1 + \frac{1}{2} ay\ t_1^2$<br>$\boxed{H = \frac{u^2\sin^2\theta}{2g}}$ |
| **time of ascent and time of descent**<br>At top most point $v_y = 0$<br>$v_y = u_y + a_y t$<br>$\Rightarrow 0 = u\sin\theta - gt$<br>$t_1 = \frac{u\sin\theta}{g}$<br>$t_2 = T - t_1 = \frac{u\sin\theta}{g}$<br>$\boxed{t_1 = t_2 = \frac{T}{2} = \frac{u\sin\theta}{g}}$ | Method 2 : using third equation of motion<br>$v_y^2 - u_y^2 = 2a_y s_y$<br>$0 - u^2\sin^2\theta = -2g\, s_y$<br>$= u\sin\theta \times \frac{u\sin\theta}{g} - \frac{1}{2}g\frac{u^2\sin^2\theta}{g^2}$<br>$\boxed{H = \frac{u^2\sin^2\theta}{2g}}$ |

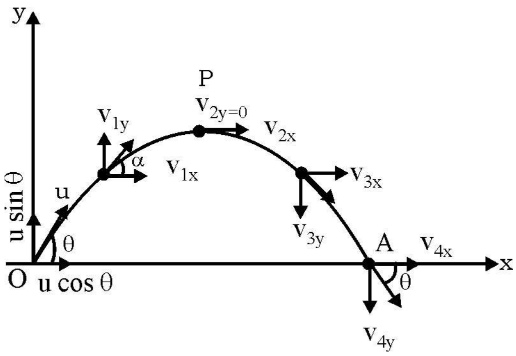
**Maximum Range**

$$R = \frac{u^2 \sin 2\theta}{g} \text{ and } R_{max} = \frac{u^2}{g}$$

Range is maximum when sin 2θ is maximum

max (sin 2θ) = 1 or, θ = 45°

### 12.1 Analysis of velocity in case of a projectile




From the above equations;

(i) $v_{1x} = v_{2x} = v_{3x} = v_{4x} = u_x = u \cos\theta$

which means that the velocity along x axis remains constant

[as there is no external force acting along that direction]

(ii) a) magnitude of velocity along y axis first decreases and then it increases after the top most point

b) at top most point magnitude of velocity is zero.

c) direction of velocity is in the upward direction while ascending and is in the downward direction while descending.

d) magnitude of velocity at A is same as magnitude of velcoity at 0; but the direction is inverse

e) angle which the net velocity makes with the horizontal can be calculated by

$$\tan\alpha = \frac{v_y}{v_x} = \frac{\text{velocity along y axis}}{\text{velocity along x axis}}$$

net velocity is always along the tangent

### 12.2 Equation of trajectory

Trajectory is the path traced by the body. To find the trajectory we must find relation between y and x by eliminating time.

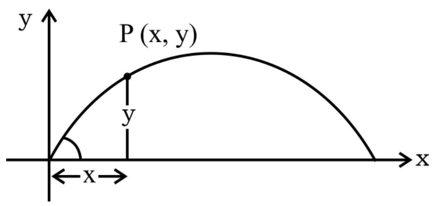
[Ref. to the earlier diag]

| Horizontal Motion | Vertical Motion |
|---|---|
| $u_x = u\cos\theta$ | $u_y = u\sin\theta$ |
| $a_x = 0$ | $a_y = -g$ |
| $s_x = u\cos\theta\, t = x$ | $s_y = u_y t + \frac{1}{2} a_y t^2$ |
| $t = \frac{x}{u\cos\theta}$ | $y = u\sin\theta\left(\frac{x}{u\cos\theta}\right) - \frac{1}{2} g \frac{x^2}{u^2\cos^2\theta}$ |

$$\boxed{\mathbf{y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta} \Rightarrow y = bx - ax^2}}$$

(i) This is a equation of a parabola

(ii) Because the co–officient of $x^2$ is negative, it is an inverted parabola.




Path of the projectile is a parabola

$$R = \frac{2u^2\sin\theta\cos\theta}{g} \quad \text{or,} \quad \frac{2u^2}{g} = \frac{R}{\sin\theta\cos\theta}$$

Substituting this value in the above equation we have,

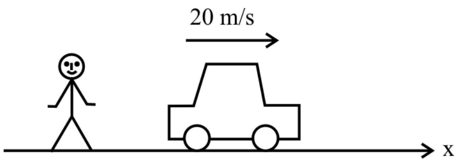
$$\boxed{\mathbf{y = x\tan\theta\left[1 - \frac{x}{R}\right]}}$$

## 13. RELATIVE MOTION

Relative is a very general term

In physics we use relative very oftenly.

For eg




**Case I :** If you are observing a car moving on a straight road then you say velocity of car is 20 m/s which means velocity of car relative to you is 20 m/s or, velocity of car relative to ground is 20 m/s (as you are standing on the ground.

**Case II :** If you go inside a car and observe you will find that the car is at rest while the road is moving back wards. you will say;

velocity of car relative to the car is 0 m/s

Mathematically, velocity of B relative to A is represented as

$$\boxed{\vec{v}_{BA} = \vec{v}_B - \vec{v}_A}$$

This being a vector quantity direction is very important

$\therefore$ $$\boxed{\vec{v}_{BA} \neq \vec{v}_{AB}}$$

## 14. RIVER–BOAT PROBLEMS

In river–boat problems we come across the following three terms :

$\vec{v}_r =$ absolute velocity of river.

$\vec{v}_{br} =$ velocity of boatman with respect to river or velocity of boatman in still water and $\vec{v}_b =$ absolute velocity of boatman.

Hence, it is important to note that $\vec{v}_{br}$ is the velocity of boatman with which he steers and $\vec{v}_b$ is the actual velocity of boatman relative to ground. Further $\vec{v}_b = \vec{v}_{br} + \vec{v}_r$

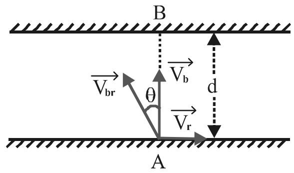
Now, let us derive some standard results and their special cases.

A boatman starts from point A on one bank of a river with velocity $\vec{v}_{br}$ in the direction shown in figure. River is flowing along positive x–direction with velocity $\vec{v}_r$. Width of the river is d. Then

$$\vec{v}_b = \vec{v}_r + \vec{v}_{br}$$

Therefore, $v_{bx} = v_{rx} + v_{brx} = v_r - v_{br} \sin\theta$

and $v_{by} = v_{by} + v_{bry} = 0 + v_{br}\cos\theta = v_{br}\cos\theta$




Now, time taken by the boatman to cross the river is :

$$\boxed{t = \frac{d}{v_{by}} = \frac{d}{v_{br}\cos\theta} \text{ or } t = \frac{d}{v_{br}\cos\theta}} \quad \text{...(i)}$$

Further, displacement along x–axis when he reaches on the other bank (also called drift) is

$$\boxed{x = v_{bx}\ t = (v_r - v_{br}\sin\theta)\frac{d}{v_{br}\cos\theta}}$$

or $$x = (v_r - v_{br}\sin\theta)\frac{d}{v_{br}\cos\theta} \quad \text{...(ii)}$$

**Condition when the boatman crosses the river in shortest interval of time**

From eq. (i) we can see that time (t) will be minimum when $\theta = 0°$ i.e., the boatman should steer his boat perpendicular to the river current.

**Condition when the boat wants to reach point B, i.e., at a point just opposite from where he started (shortest distance)**

In this case, the drift (x) should be zero.

$\therefore$ $x = 0$

or $$(v_r - v_{br}\sin\theta)\frac{d}{v_{br}\cos\theta} = 0 \text{ or } v_r = v_{br}\sin\theta$$

or $$\sin\theta = \frac{v_r}{v_{br}} \text{ or } \theta = \sin^{-1}\left(\frac{v_r}{v_{br}}\right)$$

Hence, to reach point B the boatman should row at an angle $\theta = \sin^{-1}\left(\frac{v_r}{v_{br}}\right)$ upstream from AB.

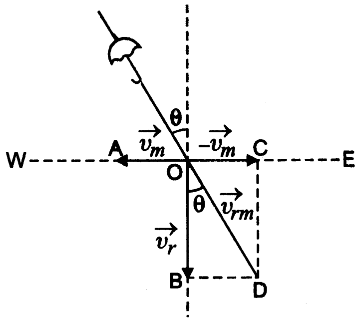
$$\boxed{t = \frac{d}{v_b} = \frac{d}{\sqrt{v_{br}^2 - v_r^2}}}$$

Since $\sin\theta \not> 1$. So, if $v_r \geq v_{br}$, the boatman can never reach at point B. Because if $v_r = v_{br}$, $\sin\theta = 1$ or $\theta = 90°$ and it is just impossible to reach at B if $\theta = 90°$. Similarly, if $v_r > v_{br}$, $\sin\theta > 1$, i.e., no such angle exists. Practically it can be realized in this manner that it is not possible to reach at B if river velocity ($v_r$) is too high.

## 15. RELATIVE VELOCITY OF RAIN W.R.T THE MOVING MAN

Consider a man walking west with velocity $\vec{v}_m$, represented by $\overrightarrow{OA}$. Let the rain be falling vertically downwards with velocity $\vec{v}_r$, represented by $\overrightarrow{OB}$. Figure. To find the relative velocity of rain with respect to man ( i.e. $\vec{v}_{rm}$) bring the man at rest by imposing a velocity $-\vec{v}_m$ on man and apply this velocity on rain also. Now the relative velocity of rain with respect to man will be the resultant velocity of $\vec{v}_r\left(=\overrightarrow{OB}\right)$ and $-\vec{v}_m\left(=\overrightarrow{OC}\right)$, which will be represented by diagonal $\overrightarrow{OD}$ of rectangle OBDC.

$$\therefore \quad v_{rm} = \sqrt{v_r^2 + v_m^2 + 2v_r v_m \cos 90°} = \sqrt{v_r^2 + v_m^2}$$




If $\theta$ is the angle which $\vec{v}_{rm}$ makes with the vertical direction then

$$\tan\theta = \frac{BD}{OB} = \frac{v_m}{v_r} \text{ or } \theta = \tan^{-1}\left(\frac{v_m}{v_r}\right)$$

Here, angle $\theta$ is from vertical towards west and is written as $\theta$, west of vertical.

**Note :** In the above problem if the man wants to protect himself from the rain, he should hold his umbrella in the direction of relative velocity of rain with respect to man i.e. the umbrella should be held making an angle $\theta$ (= $\tan^{-1} v_m/v_r$) west of vertical.